Contemplating Cam's Cake Cuts – Facilitation Guide

Yudhi Bunjamin and Louis Yudowitz

1 Introduction

This workshop is intended to get participants used to dealing with geometric problems, as well as to better appreciate how to come up with a proof (that is, think like a mathematician). Moreover, while the problems are presented as dealing with area, the solutions (at least the ones we have come up with so far) revolve around comparing the lengths of line segments. More technically speaking, the participants will see a situation where a lower dimensional quantity determines a higher dimensional one.

The precise mechanism the workshop uses to do this is a cake cutting problem. First, some warm-up tasks are assigned to the participants to get them thinking about how they can cut a rectangular or triangular cake into slices of equal area. That is, so each slice has the same amount of cake. They are then presented with the main task: determining if the areas of some other slices (which are not standard polygons!) have the same area or not. This involves understanding not only how the slices can be put together to reform the original cake, but also how their areas and side lengths relate to those of the other slices.

On the more conceptual side, the workshop relies on the fact that every polygon can be triangulated, as well as some Euclidean geometry related to bisecting and trisecting triangles into smaller triangles of equal area.

2 Logistics and Manipulatives

The workshop is intended to involve the following materials, though they can be modified according to where the workshop is being run.

- Paper
- Pens/pencils
- Warm-up sheets: some with rectangles on them, and others with triangles (see the appendix for an example)

- Cake slice pieces
- Dry Erase Markers
- Transparencies with an outline of the assembled cake
- Straightedges (unmarked!)

Currently, the workshop is designed to run for about an hour (plus time to wrap up and present solutions) with a ratio of one helper/facilitator for every four students. This may not be feasible in many situations, so future development of the workshop should try to relax this constraint.

2.1 Ideas For Longer Workshops and/or Running it with Fewer Facilitators

- In the past, we have tried having occasional intermissions where we can give students some ideas, as well as have people say what they've tried and had success with. Having small group discussions during these (and when presenting the full solution) would be great! This slows things down, but if we have more than an hour for the workshop the pacing becomes a much smaller issue.
- The 1:4 ratio was determined to make sure everyone solves or is close to solving the problem by the end. Again, this was with a 1 hour time limit in mind. It also made it so each facilitator could essentially stick with one group for the whole workshop. However, making the activity more interactive or longer would heavily reduce the need for this ratio.
- This would all probably make the activity more of a guided/group exercise. This is not necessarily a bad thing and it would also keep the students engaged. Getting the students to work in groups would help too and allow the facilitators can move around the room to talk with/help more people.

Proposed Schedule (feel free to use your own judgment though, the suggested times are just suggestions):

1. Warm-up tasks 1 and 2 (see below). Try to get students to work together on this.
2. Go over potential answers and possibly get students to come up to the board or discuss as a group.
3. Introduce the main task and let the students try on their own.
4. Try to make sure they all reassemble the cake within 10 minutes.

5. After 20-25 (rough estimate) minutes, go over how to show the yellow and green pieces are each a quarter of the cake.

6. Warm-up tasks 3 and 4 (same procedure here as earlier).

7. Let students work and guide them until 20 or so minutes are left.

8. Go over solutions (possibly only 1 for the MS students).

In tabular form:

	Suggested timing for MS	Suggested timing for HS
Warm-up 1	5	5
Go over answers	5	5
Warm-up 2	5	5
Go over answers	5	5
Main task	20	20
Interruption (yellow and green slices)	10	10
Warm-ups 3 and 4	10	10
Continue main task	20	30
Wrap-up and group discussion	10	20

3 Workshop Description

Warm-up activity

Cam likes having friends over on their birthday for cake. Each year, Cam cuts the cake differently. The warm-up task for the participants is to replicate this by cutting either squares or triangles into a set number and type of slices, each with the same area. More precisely:

1. Cut a rectangular cake into 4 rectangular pieces with equal area. See Figure 1.

2. Cut a rectangular cake into 4 triangular pieces with equal area. An important thing to note is that the slices *do not* need to be congruent! See Figure 2.

3. Cut an isosceles triangle into 2 triangles with equal area. See Figure 3.

4. Cut a scalene triangle into 3 triangles with equal area. See Figure 4.

Some notes on the warm-up and possible solutions:

- The idea in the first three warm-up tasks is to cut things in half (that is, bisect them) over and over again, in different ways each time.

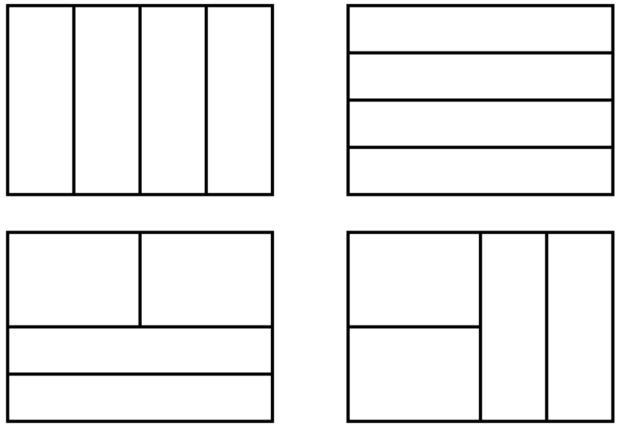

Figure 1: Some examples for the first part of the warm-up.

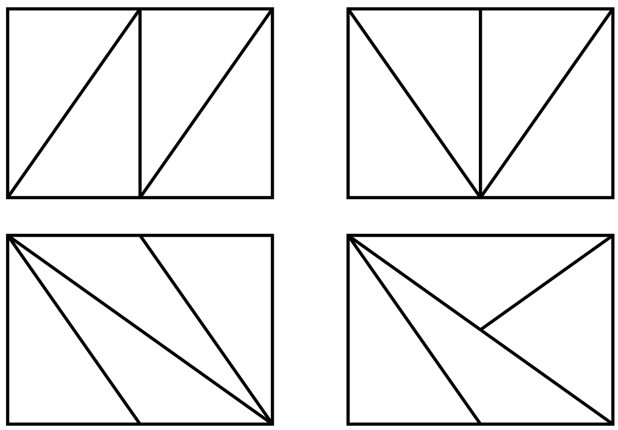

Figure 2: Some examples for the second part of the warm-up.

- For the final task, there are two approaches.

    1. One can trisect the base of the triangle and then draw a line from the points used to trisect the base to the vertex.
    2. Cut the base into a $1:2$ ratio, and then divide the right hand part in half as in Figure 4.

The New Cake

We now transition to the next stage of the workshop and present the main problem the participants should solve. A rough description of the task is the following:

Now Cam wants to shake things up and cut this year's cake in an inventive way. These new slices look stranger than usual, and his friends aren't sure that they all have the same amount of cake. All they have to try and measure things are unmarked straightedges, some pens, and sheets of paper. Your task now is to use the tools provided to show if each slice of

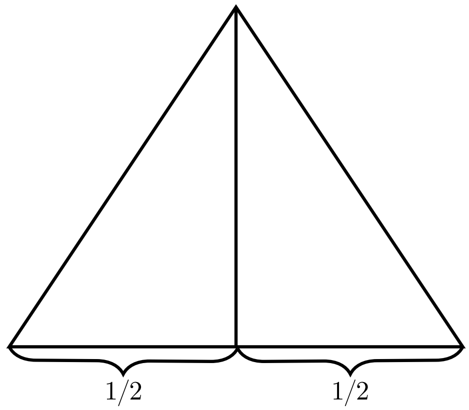

Figure 3: Solution to the third part of the warm-up.

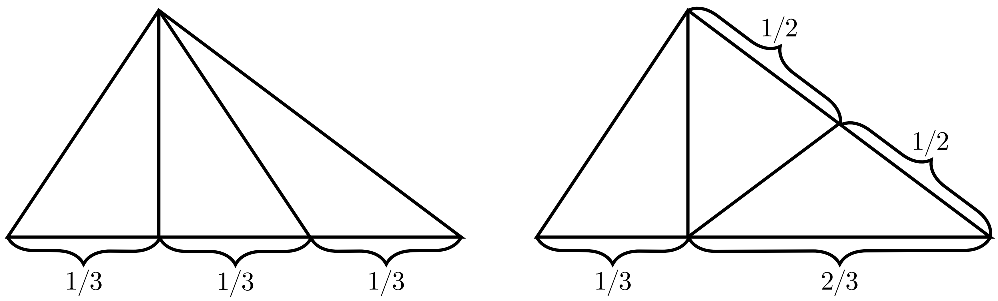

Figure 4: Possible solutions to the fourth part of the warm-up.

cake has same amount of cake (that is, the same area) as the others or if some have more or less cake.

The slices can be found in Figure 5, where they numbered for convenience when discussing them.

The colored sides are the top of each slice. There is no need to flip them over!

Depending on the presentation style, it can be *very important* that the presenter says "show each slice has the same area" and NOT "measure the area of each slice". Later on the participants should tend toward considering proportions instead of area, so establishing they should measure, rather than compare, areas helps enforce this. Moreover, if we tell them explicitly to measure area, this could lead to some participants getting angry that we are deliberately obscuring how to solve the problem if we explicitly tell them to measure areas but guide them away from it down the line.

4 Notes for Facilitators

- Given past play testing, it is a good idea for each facilitator to try to focus on helping one or two tables only (again, the $1:4$ ratio can cause issues...).

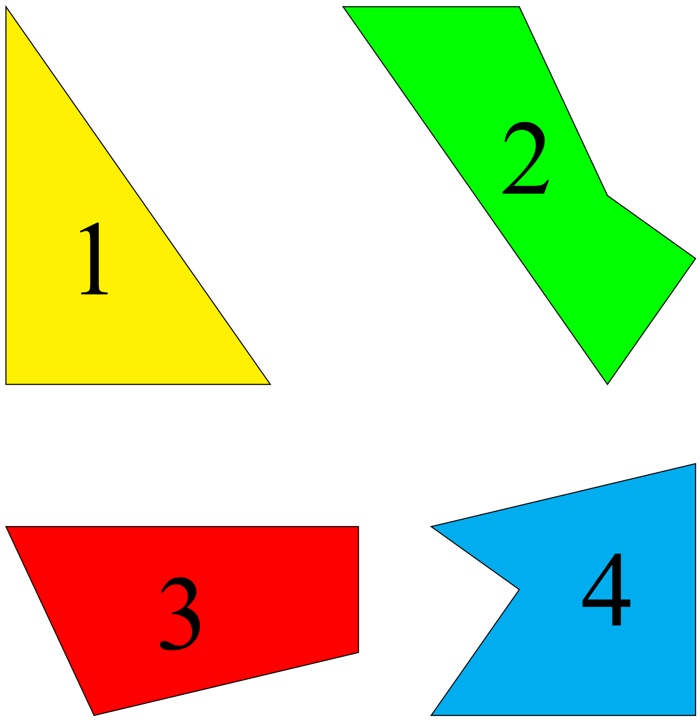

Figure 5: The cake slices.

- The main problem is also tricky, so try to encourage the participants as much as possible (again, this comes from some experiences when play testing).
- During the warm-up, it's fine if they want to experiment after finding a couple solutions.
- Usually people reassemble the cake within 5 to 7 minutes. Feel free to nudge them toward this after that time has passed. After they have reassembled the cake give them an printout with an outline of the cake on it (see Figure 4).
- *Please do not* give a printout with an outline of the cake to a participant until they have reassembled the cake.
- The most common solutions people gravitate toward end up involving exploiting $1 : 2$ ratios present in the cake (see Approaches 1 and 2 in the Solutions section). To find these, the participants should need to move the cake slices around after they have reassembled the cake. They usually seem hesitant to do this, so they might require some prompting. Handing out printouts of the reassembled cake is intended to help participants feel more comfortable

moving the pieces around after reforming the cake. It would thus be useful to remind them of this if necessary.

- As an aside, if they somehow finish early, then we can have the students explain the solution to each other, or try and design their own cakes and cutting techniques.

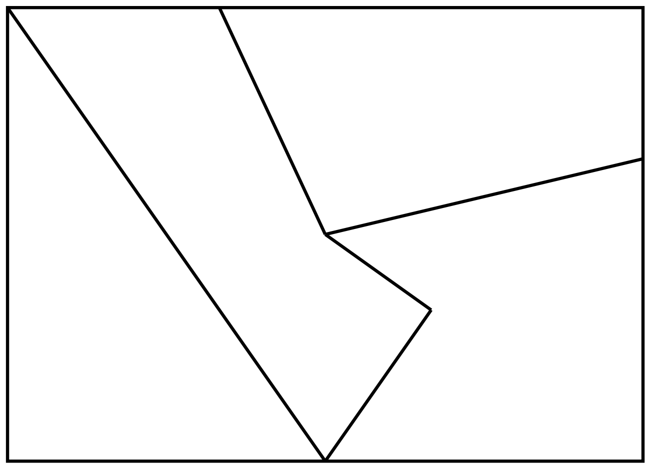

Figure 6: Outline of the reassembled cake.

5 Ending the Workshop/Wrap-Up

For the wrap-up after the participants have had time to work on and solve the main problem, we can present the most common solutions/techniques (see Solutions section). Given how complicated the diagrams can get, this will ideally need to be done slowly and thoroughly (so about 15 to 20 minutes). Of course, with a 1 hour time limit this takes up a huge portion of the event.

6 Solutions

Now on to the solutions we have found so far. Most participants in the past have gravitated toward either Approach 2 or Approach 3. Approach 1 is the most elegant, but potentially requires some luck during the warm-up and a good deal of coaching when the participants are trying to solve the main task.

Approach 1: See Figure 7 and Figure 8 for a visualization of the following solution. This (or slight variations of it) seems to be one that many people gravitate toward.

Solution steps:

1. Bisect the cake by drawing the line segments $\overline{AB}$ and $\overline{BC}$.

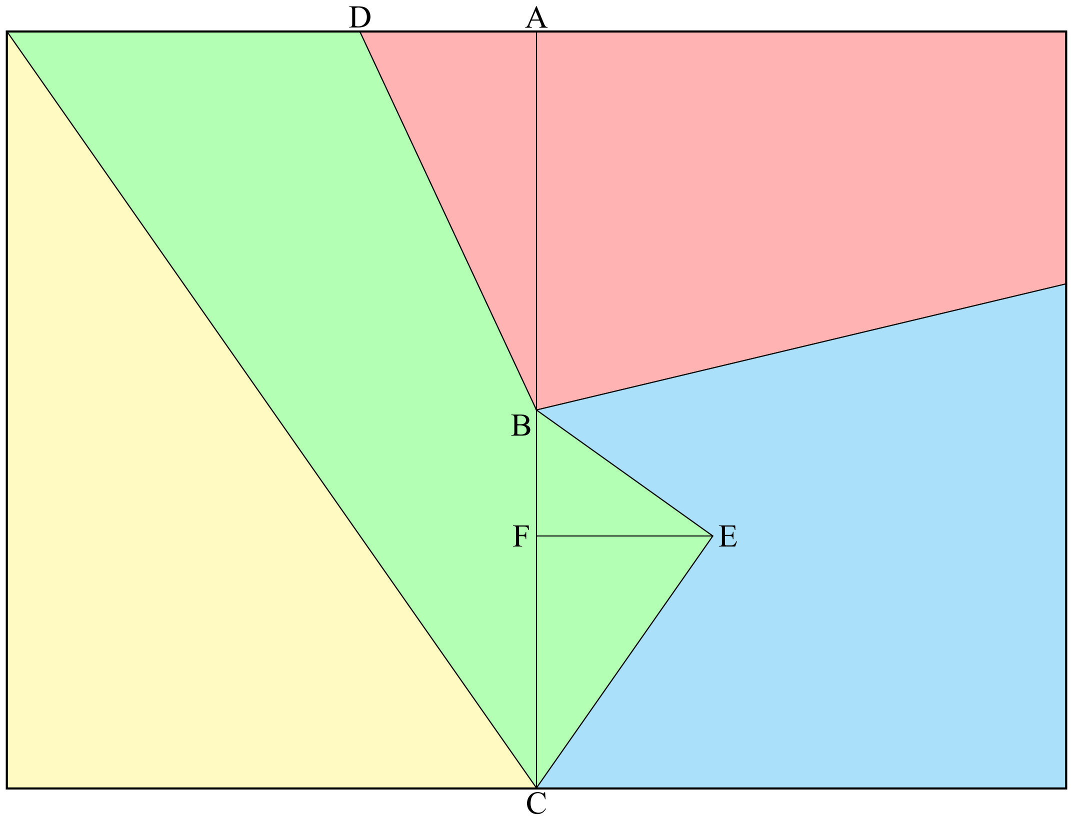

Figure 7: Visualization of Approach 1.

2. Note the yellow triangle is thus one quarter of the cake as it is one half of the left side of the bisected cake.

3. Using the straightedge (or otherwise), one can measure that

$$\text{Length}\left(\overline{AB}\right) = \text{Length}\left(\overline{BC}\right).$$

   Note this also follows from noting that B is the midpoint of the cake.

4. Also, we have

$$\text{Length}\left(\overline{AD}\right) = \text{Length}\left(\overline{EF}\right).$$

   Overlaying the red and green pieces and noting $\overline{AD}$ is the same length as $\overline{EF}$ works as well.

5. Combining the last two items and using the formula for the area of a triangle, we see that

$$\text{Area}\left(\triangle BEC\right) = \text{Area}\left(\triangle BAD\right).$$

6. Now, since the yellow triangular piece is one-quarter of the cake, we see that the green piece must be a quarter of the cake by Item 5.
7. It remains to consider the right half of the cake (refer to Figure 8 from now on).

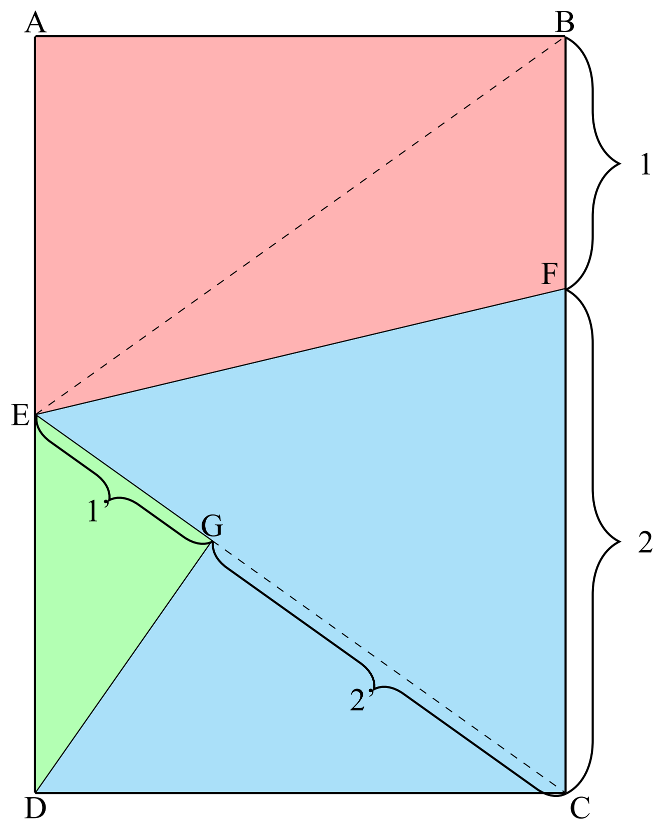

Figure 8: Continuation of the visualization of Approach 1.

8. Note that the triangle $\triangle EGD$ has the same area as the remaining part of the red piece that is left out of Figure 8.
9. The triangles $\triangle ABE$ and $\triangle EDC$ are each $\frac{1}{4}$ of the rectangle $ABCD$, while the triangle $\triangle EBC$ makes up $\frac{1}{2}$ of the area of the rectangle $ABCD$.
10. Using the straightedge we can measure that

$$\text{Length}\left(\overline{GC}\right) = 2\text{Length}\left(\overline{EG}\right)$$

Note that this can be done by overlaying the portion of the green piece corresponding to $\overline{EG}$ on the part of the blue piece corresponding to $\overline{GC}$ (and noting this needs to be done twice).
11. Therefore

$$\text{Area}\left(\triangle GDC\right) = 2\text{Area}\left(\triangle EGD\right).$$

12. Similarly, we can deduce that

$$\text{Length}\left(\overline{FC}\right) = 2\text{Length}\left(\overline{BF}\right)$$

Note that this can be done by overlaying the portion of the red piece corresponding to BF on FC (and noting this needs to be done twice).

13. Therefore,

$$\text{Area}\left(\triangle FEC\right) = 2\text{Area}\left(\triangle EBF\right).$$

14. Putting everything together, we see that

$$Area\left(Red\right) = Area\left(\triangle EGD\right) + Area\left(\triangle ABE\right) + Area\left(\triangle BEF\right) = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} = \frac{1}{2}$$

and also that

$$Area\left(Blue\right) = Area\left(\triangle GCD\right) + Area\left(\triangle FEC\right) = \frac{2}{6} + \frac{1}{6} = \frac{1}{2}$$

In other words, the blue and red pieces each makeup one-quarter of the cake.

Approach 2: See Figure 11 for a visualization of the following approach. This is a variation of Approach 1 and the only changes occur after showing the green and yellow cake slices are each one quarter of the cake.

1. Note that considering the line $\overline{EH}$ tells us if we can show $\text{Area}\left(\triangle EFH\right) = \text{Area}\left(\triangle EGD\right)$ then we are done.

2. We know that $\text{Area}\left(\triangle CEB\right) = \frac{1}{2}$, so drawing the line $\overline{EJ}$ gives us

$$\text{Area}\left(\triangle BEF\right) = \text{Area}\left(\triangle FEI\right) = \text{Area}\left(\triangle CEI\right) = \frac{1}{6}.$$

3. Since $\overline{EH}$ bisects the rectangle $ABCD$, it bisects $\triangle FEJ$.

4. Therefore, $\text{Area}\left(\triangle FEH\right) = \frac{1}{12}$. Proceeding as in Approach 2 tells us this is the area of $\triangle EGD$ and we are done!

Approach 3: The starting point is to realize the cake in Figure 10 is a valid solution to the second part of the warm-up. After this, you do the following (see Figure 11 for a visualization):

1. Draw a bisector along the main diagonal (from point A to point C).

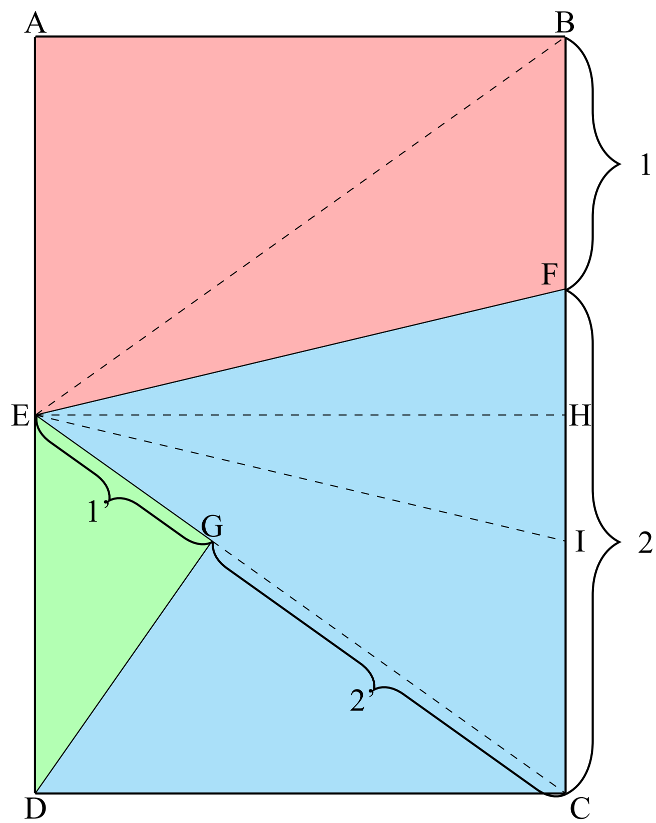

Figure 9: Visualization of Approach 2.

2. Bisect the right-hand side by line a by drawing a line from the corner B to O.

3. Split the line segment $\overline{AB}$ into thirds and connect points E and F to O.

4. Split the line segment $\overline{BC}$ into thirds and connect the points G and H to O.

5. Split the line $\overline{AC}$ into thirds and connect point I to J.

6. Split the line $\overline{AJ}$ into thirds and connect points K and L to O.

7. Now note that on each side of the line $\overline{AC}$ we have half of the cake. Moreover, the line $\overline{AJ}$ splits the left half into equal pieces while the line $\overline{BO}$ splits the right half into equal pieces.

8. By construction each quarter has been split into 3 equal pieces, hence all the smaller triangles we have made are equal (have the same area/are the same proportion of the cake).

9. Now we notice that each original slice of the cake is made up of exactly three distinct smaller triangles.

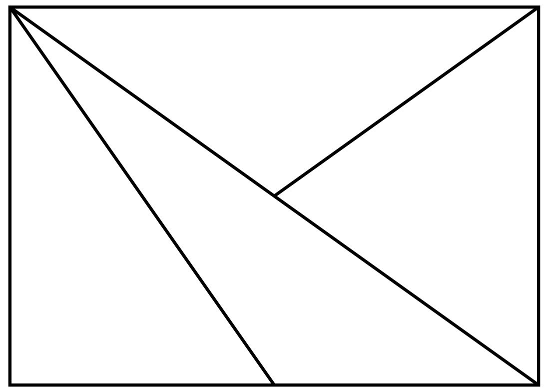

Figure 10: Important figure from the warm up.

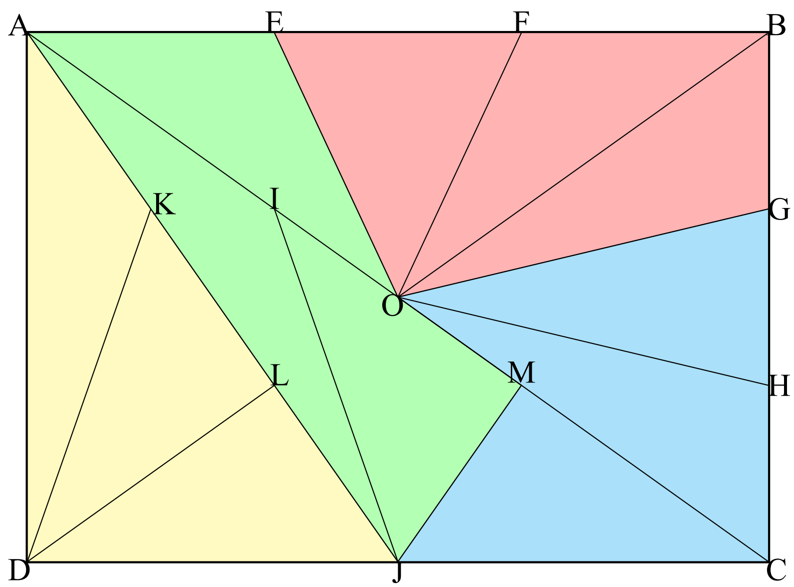

Figure 11: Visualization of Approach 3.

Approach 4: This is a rather more direct approach and does not rely on splitting a triangle into 3 triangular pieces of equal area. It does, however, use $1 : 2$ ratios in multiple ways. The final result is shown in Figure 12.

1. Split the cake into 4 rectangles AMOG, GOJB, MOKD, and OJCK, each being a quarter of the cake.
2. Then draw the line $\overline{\mathrm{EL}}$ to cut MOKD into two equal rectangles MPLD and POKL.
3. Note $\mathrm{Area}\,(MPLD) = \frac{1}{8}$, while POKL is split into two equal triangles $\triangle POK$ and $\triangle PLK$ by the line $\overline{AK}$. Thus $\mathrm{Area}\,(\triangle POK) = \frac{1}{16}$ and $\mathrm{Area}\,(\triangle PLK) = \frac{1}{16}$.
4. Similarly, $\mathrm{Area}\,(AMP) = \frac{1}{16}$ and $\mathrm{Area}\,(PEA) = \frac{1}{16}$.

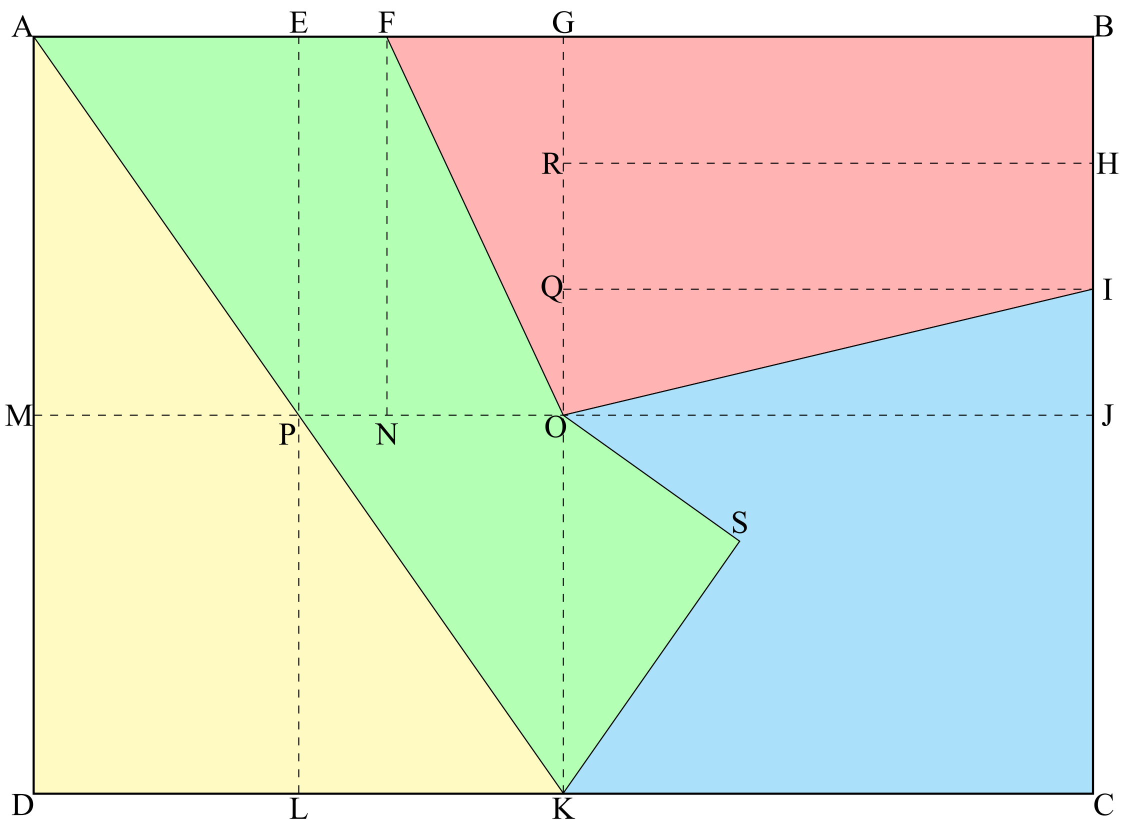

Figure 12: End result of Approach 4.

5. This tells us

$$\begin{aligned}\text{Area}\left(Yellow\right) &= \text{Area}\left(\triangle AMP\right) + \text{Area}\left(MPDL\right) + \text{Area}\left(\triangle PLK\right) \\ &= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}\end{aligned}$$

   Alternatively, one can deduce the yellow slice is $\frac{1}{4}$ as in other approaches (which is probably more straightforward).

6. Comparing the physical slices, we can deduce that $\text{Length}\left(\overline{FB}\right) = 2\text{Length}\left(\overline{AF}\right)$.

7. Since $\overline{GK}$ splits the cake ABCD in half, we can infer that $\text{Length}\left(\overline{FG}\right) = 2\text{Length}\left(\overline{EF}\right)$.

8. The previous step means that $\text{Area}\left(EFNP\right) = \frac{1}{24}$ of the cake. Additionally, $\text{Area}\left(FON\right) = \frac{1}{24}$ and $\text{Area}\left(FGO\right) = \frac{1}{24}$.

9. Note that we can deduce that $\triangle OSK$ and $\triangle FGO$ have the same area as usual.

10. Then we can tell that

$$
\begin{aligned}
&\text{Area}\,(Green) \\
&= \text{Area}\,(\triangle AEP) + \text{Area}\,(EFPN) + \text{Area}\,(\triangle FON) + \text{Area}\,(\triangle POK) + \text{Area}\,(\triangle OSK) \\
&= \frac{1}{16} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} + \frac{1}{24} \\
&= \frac{1}{4}.
\end{aligned}
$$

11. Now split the rectangle GOJB into thirds by drawing the lines $\overline{QI}$ and $\overline{RH}$.

12. Comparing the blue and red slices, we know that $\text{Length}\left(\overline{CI}\right) = 2\text{Length}\left(\overline{IB}\right)$.

13. Putting the last two steps together, we can deduce that $\text{Area}\,(GBRH) = \text{Area}\,(RHIQ) = \frac{1}{12}$, while $\text{Area}\,(QIO) = \frac{1}{24}$.

14. Then

$$
\begin{aligned}
\text{Area}\,(Red) &= \text{Area}\,(\triangle FOG) + \text{Area}\,(GBRH) + \text{Area}\,(RHIQ) + \text{Area}\,(QIO) \\
&= \frac{1}{24} + \frac{1}{12} + \frac{1}{12} + \frac{1}{24} \\
&= \frac{1}{4}.
\end{aligned}
$$

7 Appendix: Example of a Warm-Up Handout